

The Aharonov-Bohm Effect in Noncommutative Quantum Mechanics

Kang Li^{a,*} and Sayipjamal Dulat^{b,†}

^a*Department of Physics, Hangzhou Teachers College, Hangzhou, 310036, China*

^b*Department of Physics, Xinjiang University, Urumqi, 830046, China and*

^c*The Abdus Salam International Center for Theoretical Physics, Trieste, Italy*

The Aharonov-Bohm (AB) effect in non-commutative quantum mechanics (NCQM) is studied. First, by introducing a shift for the magnetic vector potential we give the Schrödinger equations in the presence of a magnetic field on NC space and NC phase space, respectively. Then by solving the Schrödinger equations, we obtain the Aharonov-Bohm (AB) phase on NC space and NC phase space, respectively.

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INTRODUCTION: NC SPACE AND NC PHASE SPACE

Recently, there has been much interest in the study of physics on noncommutative (NC) space [1]-[6], not only because the NC space is necessary when one studies the low energy effective theory of D-brane with B field background, but also because in the very tiny string scale or at very high energy situation, the effects of non commutativity of both space-space and momentum-momentum may appear. There are many papers devoted to the study of various aspects of quantum mechanics [7]-[34] on noncommutative space with usual (commutative) time coordinate.

In the usual n dimensional commutative space, the coordinates and momenta in quantum mechanics have the following commutation relations:

$$\begin{aligned} [x_i, x_j] &= 0, \\ [p_i, p_j] &= 0, \\ [x_i, p_j] &= i\hbar\delta_{ij}. \end{aligned} \quad i, j = 1, 2, \dots, n, \quad (1)$$

At very tiny scales, say string scale, not only space-momentum does not commute, but also space-space may not commute anymore. Therefore the NC space is a space where coordinate and momentum operators satisfy the following commutation relations

$$[\hat{x}_i, \hat{x}_j] = i\Theta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad (2)$$

where \hat{x}_i and \hat{p}_i are the coordinate and momentum operators on NC space. Ref.[34] showed that $\hat{p}_i = p_i$ and \hat{x}_i have the representation form

$$\hat{x}_i = x_i - \frac{1}{2\hbar}\Theta_{ij}p_j, \quad i, j = 1, 2, \dots, n. \quad (3)$$

The case of both space-space and momentum-momentum noncommuting [30][34] is different from the case of only space-space noncommuting. Thus the NC phase space is a space where momentum operator in Eq. (2) satisfies the following commutation relations

$$[\hat{p}_i, \hat{p}_j] = i\bar{\Theta}_{ij}, \quad i, j = 1, 2, \dots, n. \quad (4)$$

Here $\{\Theta_{ij}\}$ and $\{\bar{\Theta}_{ij}\}$ are totally antisymmetric matrices which represent the noncommutative property of the coordinate and momentum on noncommutative space and phase space, respectively, and play analogous role to \hbar in the usual quantum mechanics. On NC phase space the representations of \hat{x} and \hat{p} in term of x and p were given in ref.[34] as follows

$$\begin{aligned} \hat{x}_i &= \alpha x_i - \frac{1}{2\hbar\alpha}\Theta_{ij}p_j, \\ \hat{p}_i &= \alpha p_i + \frac{1}{2\hbar\alpha}\bar{\Theta}_{ij}x_j, \end{aligned} \quad i, j = 1, 2, \dots, n. \quad (5)$$

The α here is a scaling constant related to the noncommutativity of phase space. When $\bar{\Theta} = 0$, it leads $\alpha = 1$ [34], the NC phase space returns to the NC space, which is extensively studied in the literature, where the space-space is non-commuting, while momentum-momentum is commuting.

Given the NC space or NC phase space, one should study its physical consequences. It appears that the most natural places to search the noncommutativity effects are simple quantum mechanics (QM) system. So far many

interesting topics in NCQM such as hydrogen atom spectrum in an external magnetic field [12, 15] and Aharonov-Bohm(AB) effect [14] in the presence of the magnetic field, as well as the Aharonov-Casher effects [16] have been studied extensively. The purpose of this paper is to do further study on the Aharonov-Bohm effect on NC space and NC phase space, respectively, where the space-space noncommutativity and both space-space and momentum-momentum noncommutativity could produce additional phase difference.

This paper is organized as follows: In section 2, we study the Aharonov-Bohm effect on NC space. First, the Schrödinger equation in the presence of magnetic field is given, and the magnetic Aharonov-Bohm phase expression is derived. In two dimensions, our result agrees with the result of Ref. [14]. The general AB phase on NC space is also given in the presence of electromagnetic field. In section 3, we investigate the Aharonov-Bohm effect on NC phase space. By solving the Schrödinger equation in the presence of magnetic field, the additional AB phase related to the momentum-momentum noncommutativity is obtained explicitly. Conclusions and some remarks are given in section 4.

THE AHARONOV-BOHM EFFECT ON NC SPACE

Let $H(x, p)$ be the Hamiltonian operator of the usual quantum system, then the static Schrödinger equation on NC space is usually written as

$$H(x, p) * \psi = E\psi, \quad (6)$$

where the Moyal-Weyl (or star) product between two functions is defined by,

$$(f * g)(x) = e^{\frac{i}{2}\Theta_{ij}\partial_{x_i}\partial_{x_j}} f(x_i)g(x_j) = f(x)g(x) + \frac{i}{2}\Theta_{ij}\partial_i f \partial_j g|_{x_i=x_j}, \quad (7)$$

here $f(x)$ and $g(x)$ are two arbitrary functions. On NC space the star product can be replaced by a Bopp's shift [4], i.e. the star product can be changed into the ordinary product by replacing $H(x, p)$ with $H(\hat{x}, \hat{p})$. Thus the Schrödinger equation can be written as,

$$H(\hat{x}_i, \hat{p}_i)\psi = H(x_i - \frac{1}{2\hbar}\Theta_{ij}p_j, p_i)\psi = E\psi. \quad (8)$$

Here x_i and p_i are coordinate and momentum operators in usual quantum mechanics. Thus the Eq.(8) is actually defined on commutative space, and the noncommutative effects can be evaluated through the Θ related terms. Note that the Θ term always can be treated as a perturbation in QM, since $\Theta_{ij} \ll 1$.

When magnetic field is involved, the Schrödinger equation (6) becomes

$$H(x_i, p_i, A_i) * \psi = E\psi. \quad (9)$$

To replace the star product in Eq.(9) with a usual product, first we need to replace x_i and p_i with a Bopp's shift, then we also need to replace the vector potential A_i with a shift given as follows

$$A_i \rightarrow A_i + \frac{1}{2}\Theta_{lj}p_l\partial_j A_i, \quad (10)$$

Thus the Schrödinger Eq.(9) in the presence of magnetic field becomes

$$H(x_i - \frac{1}{2\hbar}\Theta_{ij}p_j, p_i, A_i + \frac{1}{2}\Theta_{lj}p_l\partial_j A_i)\psi = E\psi. \quad (11)$$

We should emphasize that the Bopp's shift and the shift Eq. (10) are equivalent to the star product in the Schrödinger Eq.(9).

Now let us consider a particle of mass m and charge q moving in a magnetic field with magnetic potential A_i , then the Schrödinger equation is (we choose unit of $\hbar = c = 1$),

$$\frac{1}{2m}\left(p_i - qA_i - \frac{1}{2}q\Theta_{lj}p_l\partial_j A_i\right)^2 \psi = E\psi. \quad (12)$$

In an analogous way as in usual quantum mechanics, the solution for (12) reads

$$\psi = \psi_0 \exp \left[iq \int_{x_0}^x \left(A_i + \frac{1}{2} \Theta_{lj} p_l \partial_j A_i \right) dx_i \right], \quad (13)$$

where ψ_0 is the solution of (12) when $A_i = 0$. The phase term of (13) is so called AB phase. If we consider a charged particle to pass a double slits, then the integral runs from the source x_0 through one of the two slits to the screen x , the coherent pattern will depend on the phase difference of two paths. Thus the total phase shift for the AB effect is

$$\Delta \Phi_{AB} = \delta \Phi_0 + \delta \Phi_{\theta}^{NC} = iq \oint A_i dx_i + \frac{iq}{2} \oint \Theta_{lj} (mv_l + qA_l) \partial_j A_i dx_i, \quad (14)$$

where the relation $mv_l = p_l - qA_l + O(\Theta)$ has been applied¹, and we omitted the second order terms of the Θ ; the first term is the AB phase in usual quantum mechanics, the second term is the correction to the usual AB phase due to space-space noncommutativity; the line integral runs from the source through one of the two slits to the screen and returns to the source through the other slit.

In three dimensional NC space, i.e. $i, j = 1, 2, 3$, we can define a vector $\theta = (\theta_1, \theta_2, \theta_3)$ with θ_i satisfies $\Theta_{ij} = \epsilon_{ijk} \theta_k$, or $\theta_i = \frac{1}{2} \epsilon_{ijk} \Theta_{jk}$. Then the second and third terms in Eq.(14) have the form

$$\frac{i}{2} q \oint \Theta_{lj} mv_l \partial_j A_i dx_i = \frac{i}{2} q \oint \epsilon_{lmk} \theta_k mv_l \partial_j A_i dx_i = \frac{i}{2} qm \oint \theta \cdot (\mathbf{v} \times \nabla A_i) dx_i, \quad (15)$$

and

$$\frac{i}{2} q^2 \oint \Theta_{lj} A_l \partial_j A_i dx_i = \frac{i}{2} q^2 \oint \epsilon_{lmk} \theta_k A_l \partial_j A_i dx_i = \frac{i}{2} q^2 \oint \theta \cdot (\mathbf{A} \times \nabla A_i) dx_i. \quad (16)$$

Using Eqs. (15) and (16), we can write the AB phase as

$$\Delta \Phi_{AB} = iq \oint A_i dx_i + \frac{i}{2} q \oint (m\theta \cdot (\mathbf{v} \times \nabla A_i) + q\theta \cdot (\mathbf{A} \times \nabla A_i)) dx_i. \quad (17)$$

In two dimensional NC plane ($i, j = 1, 2$), if we consider an electron ($q = -e$) moving in a magnetic field, then the vector θ defined above just has the third component θ_3 and $\Theta_{ij} = \theta_3 \epsilon_{ij}$, $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$, then we have

$$\Delta \Phi_{AB} = -ie \oint A_i dx_i - \frac{i}{2} e\theta_3 \oint (m(\mathbf{v} \times \nabla A_i)_3 - e(\mathbf{A} \times \nabla A_i)_3) dx_i. \quad (18)$$

We should emphasize that, in two dimensional NC plane, our result (18) is exactly the same as in Ref. [14].

The AB phase expression (14) give us a hint that when a charged particle moves in an electromagnetic field with four dimensional potential A_μ , then the corresponding AB phase will have the following general expression,

$$\Delta \Phi_{AB} = iq \oint (A_\mu + \frac{1}{2} \Theta_{\alpha\beta} (mv_\alpha + qA_\alpha) \partial_\beta A_\mu) dx^\mu. \quad (19)$$

The second term is the consequence of space-space non-commutativity.

THE AHARONOV-BOHM EFFECT ON NC PHASE SPACE

The Bose-Einstein statistics in NCQM requires both space-space and momentum-momentum non-commutativity. Thus we should also consider the momentum-momentum non-commutativity when we deal with physical problems.

¹ By Eq.(12), one writes the velocity operator on NC space as $v_l = \frac{\partial H}{\partial p_l} = \frac{1}{m} (p_l - qA_l - \frac{1}{2} q \Theta_{ij} p_i \partial_j A_l - \frac{1}{2} q \Theta_{lj} (p_i - qA_i) \partial_j A_i + O(\Theta^2)) = \frac{1}{m} (p_l - qA_l + O(\Theta))$.

On NC phase space the non-commuting coordinates \hat{x}_i and momentum \hat{p}_i were given in Eq. (5). On NC phase space the star product in Eqs. (7) becomes,

$$(f * g)(x, p) = e^{\frac{i}{2\alpha^2}\Theta_{ij}\partial_i^x\partial_j^x + \frac{i}{2\alpha^2}\bar{\Theta}_{ij}\partial_i^p\partial_j^p} f(x, p)g(x, p) = f(x, p)g(x, p) + \frac{i}{2\alpha^2}\Theta_{ij}\partial_i^x f \partial_j^x g|_{x_i=x_j} + \frac{i}{2\alpha^2}\bar{\Theta}_{ij}\partial_i^p f \partial_j^p g|_{p_i=p_j}. \quad (20)$$

To replace the star product in Schrödinger Eq.(9) with a usual product, first we need to replace x_i and p_i with a generalized Bopp's shift as

$$\begin{aligned} x_i &\rightarrow x_i - \frac{1}{2\hbar\alpha^2}\Theta_{ij}p_j, \\ p_i &\rightarrow p_i + \frac{1}{2\hbar\alpha^2}\bar{\Theta}_{ij}x_j, \end{aligned} \quad (21)$$

and then we need to replace A_i with the generalized shift as

$$A_i \rightarrow \alpha A_i + \frac{1}{2\alpha}\Theta_{lj}p_l\partial_j A_i. \quad (22)$$

Thus on NC phase space the Schrödinger Eq.(9) becomes,

$$H\left(x_i - \frac{1}{2\hbar\alpha^2}\Theta_{ij}p_j, p_i + \frac{1}{2\hbar\alpha^2}\bar{\Theta}_{ij}x_j, A_i + \frac{1}{2\alpha}\Theta_{lj}p_l\partial_j A_i\right)\psi = E\psi. \quad (23)$$

One may note that the Eq.(21) is different from the Eq.(5), by Eq.(21), the other physical quantities may also be shifted, for example, mass may be replaced with $m \rightarrow m/\alpha^2$ and the electric charge q may be replaced with q/α .

Here, again, we consider a particle of mass m and electric charge q moving in a magnetic field. On NC phase space, the Hamiltonian have the form,

$$\begin{aligned} \hat{H} &= \frac{1}{2m}\left(\alpha p_i + \frac{1}{2\alpha}\bar{\Theta}_{ij}x_j - q(\alpha A_i + \frac{1}{2\alpha}\Theta_{lj}p_l\partial_j A_i)\right)^2 \\ &= \frac{1}{2m'}\left(p_i + \frac{1}{2\alpha^2}\bar{\Theta}_{ij}x_j - q(A_i + \frac{1}{2\alpha^2}\Theta_{lj}p_l\partial_j A_i)\right)^2, \end{aligned} \quad (24)$$

with $m' = m/\alpha^2$. Thus the total phase shift for the AB effect including the contribution due to both space-space and momentum-momentum non-commutativity on 3-dimensional NC phase space is ²

$$\begin{aligned} \Delta\Phi_{AB} = \delta\Phi_{NCPS} &= iq \oint A_i dx_i + \frac{iq}{2\alpha^2} \oint (m'\theta \cdot (\mathbf{v} \times \nabla A_i) + q\theta \cdot (\mathbf{A} \times \nabla A_i)) dx_i - \frac{i}{2\alpha^2} \oint \bar{\Theta}_{ij}x_j dx_i \\ &= iq \oint A_i dx_i + \frac{i}{2}q \oint (m\theta \cdot (\mathbf{v} \times \nabla A_i) + q\theta \cdot (\mathbf{A} \times \nabla A_i)) dx_i + \delta\Phi_{\bar{\theta}}^{NCPS}. \end{aligned} \quad (25)$$

Where the $\delta\Phi_{\bar{\theta}}^{NCPS}$ is the first order modification term due to momentum-momentum non-commutativity, and it has the form

$$\delta\Phi_{\bar{\theta}}^{NCPS} = -\frac{i}{2\alpha^2} \oint \bar{\Theta}_{ij}x_j dx_i + \frac{i}{2\alpha^2}(1 - \alpha^2) \oint q\theta \cdot (\mathbf{A} \times \nabla A_i) dx_i + \frac{i}{2\alpha^4}(1 - \alpha^4)q \oint (m\theta \cdot (\mathbf{v} \times \nabla A_i)) dx_i. \quad (26)$$

It is obvious from (26) that, when $\alpha = 1$, then we have $\bar{\Theta}_{ij} = 0$ as well as $\delta\Phi_{\bar{\theta}}^{NCPS} = 0$, so the AB phase returns to its expression Eq.(17) on NC space.

CONCLUSION REMARKS

In this article we study the Aharonov-Bohm effect in NCQM. The consideration of the NC space(NC phase space) produces an additional phase difference. In order to obtain the NC space correction to the usual Aharonov-Bohm

² In a similar way as in NC space, we have the relation $m'v_l = p_l - qA_l + O(\Theta) + O(\bar{\Theta})$ on NC phase space, and we omitted the second order terms of Θ and $\bar{\Theta}$ in Eq. (25).

phase difference, in section two, first, we give the Schrödinger equation in the presence of magnetic field, by solving the equation we derive the magnetic Aharonov-Bohm phase expression. Note that the non-commutative effects of the space (phase space) in the usual Schrödinger equation can be realized in two steps. First step is to replace the coordinate and momentum operators with a so called Bopp's (generalized Bopp's) shift, and then to replace the magnetic potential \mathbf{A} with a special shift which we defined in Eq.(10) in our paper. It is worth to mention that, on NC plane, our result (18) coincides with the result of Ref. [14]. In order to obtain the NC phase space correction to the usual Aharonov-Bohm phase difference, in section 3, we solve the Schrödinger equation in the presence of magnetic field and obtain the magnetic Aharonov-Bohm phase expression. Especially the new term $\delta\Phi_{\theta}^{NCPS}$ which comes from the momentum-momentum noncommutativity is given explicitly.

The method we employed in this paper may also be applied to the other related physical problems on NC space and NC phase space. For example, the Aharonov-Casher effect in NC quantum mechanics. Further study on the related topics will be reported in our forthcoming paper.

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* Electronic address: kangli@hztc.edu.cn

† Electronic address: sdulat@xju.edu.cn

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